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A Statistical Analysis of the Two-Slit Experiment: Or Some Remarks on Quantum Probability

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1. A warning first in order to avoid any misunderstanding; quantum mechanics is a great triumph of contemporary physics; moreover, it has revolutionized our understanding of the external world. Thanks to this mechanics, impredictability has become central in our approach to explain natural phenomena. I am firmly convinced of this. The statistical analysis I am going to present is not intended to show that there is something wrong in quantum mechanics. What I want to emphasize is that the probabilistic notions used in quantum mechanics deserve, from a foundational point of view, a more accurate study than what has been devoted to them.

Probability is the notion with which we try to approach unpredictability. Since the twenties, the notion of probability has assumed a central role in quantum mechanics. The early use of this notion was ambiguous. Reichenbach (1942) stands in witness of this. This author is well aware of the fact that in quantum phenomena "probabilities" interfere, but in spite of this he maintains that quantum mechanics does not make use of a special theory of probability. In his opinion, notwithstanding the way in which probability is determined, the concept of probability used in quantum mechanics is the traditional one. But less than 10 years later the notion of probability used in quantum mechanics lost any ambiguity. The point of view of Reichenbach was completely reversed by Feynman (1951). His thesis can be summarized as follows: the probability of Heisenberg, Schrödinger, Dirac, and Born is not that of Bernoulli, Bayes, Laplace, and Gauss.

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The problem I address in the present paper is strictly connected with the work of Feynman on probability. More precisely the point is this: Feynman is right in maintaining that with quantum mechanics something new has been introduced into the theory of classical probability, but I suspect that the novelty is more related to a broadening of this theory than to its falsification. In support of this suspicion, I have the arguments which I will present in this paper. In my opinion, these are at least enough to recommend a deeper analysis of the relationships between quantum and classical probabilities.

The use of the adjective "classical" needs some comments. In probability theory and statistics, when speaking of classical probability, people refer to Laplace's definition: the probability of an event is the ratio of the number of favorable cases to the whole number of possible ones. On the contrary, in quantum mechanics, when speaking of classical probability, people refer to a probability satisfying the addition rule. Feynman says: "But more fundamental was the discovery that in nature the laws combining probabilities were not those of the classical probability theory of Laplace" (Feynman, 1951, p. 533). In other words, whereas the behavior of macroscopic objects is ruled by a classical, i.e., additive probability function, the behavior of microscopic objects is ruled by a quantum, i.e., nonadditive probability function. To make the point clear, I shall use again Feynman's words: "the laws of probability which are conventionally applied are quite satisfactory in analysing the behavior of the roulette wheel but not the behavior of a single electron or a photon of light" (Feynman, 1951, p. 533).

In what follows I shall use "classical" in the same sense as intended by Feynman. Thus the suspicion I have expressed can be reformulated in this way: the laws of quantum probability are an enlargement of the classical probability theory. I am not able to make precise what this enlargement amounts to. To define exactly in what it consists is another way to pose the problem I have in mind.

2. To the best of my knowledge, Feynman's analysis of the two-slit experiment is the starting point of all reflections concerning the laws of probability in quantum mechanics. From a statistical point of view, this analysis is lacking in many aspects. The first part of my paper is devoted to justify this assertion.

As usual, I start analyzing the behavior of macroscopic particles like bullets. Following Feynman, I imagine an experiment in which a stream of bullets is being shot by a machine gun through a wall provided with two holes, 1 and 2, to a backstop capable of "absorbing" the bullets. Having done this experiment, one ascertains that the frequency of the bullets absorbed at a "point" x of the backstop when both holes are open is the sum of the frequencies of the bullets absorbed at "point" x when only hole 1 is open and when only hole 2 is open. Regarding this, Feynman asserts, "The chance of arrival at X should be the sum of two parts, P_1, the chance of arrival coming through hole 1 plus P_2, the chance of arrival coming through hole 2" (Feynman, 1951, p. 535). Due to this fact, for Feynman, the experiment with bullets ensures that in the case of macroscopic objects probabilities add together. The behavior of bullets is ruled by classical probability, in particular the addition rule holds.

This conclusion is too rough. What the experiment with bullets corroborates is not the laws of probabilities, but those ruling the theory of errors in measurement. For the sake of simplicity, I suppose that the distribution of bullets coming from one hole while the other is closed is Gaussian. If this is the case, the explanation of the distribution on the backstop can be given as follows. As is well known, the point of arrival of a bullet is supposed to result from the interaction of an infinite number of small and independent causes each of which, should it act alone, will give rise to an elementary error. Therefore, the actual point of arrival can be assumed to be a random variable being the (limit) sum of an infinite sequence of random variables each with expectation equal to zero. Each variable of the sequence is related to one of these elementary errors. If we suppose that the random variables of the sequence are independent (and that their individual variances are small as compared with their sum), then the distribution of the normalized sum of these variables is asymptotically Gaussian (with zero expectation and unit variance). This is essentially the central limit theorem as proved by Lindeberg, but it is not sufficient to explain the distribution of bullets. We must suppose the independence of random variables describing also bullets. In this way, via the law of large numbers, it is possible to explain each of the two Gaussian distributions on the backstop when only one hole is open. Finally, supposing independence of distributions resulting from arrivals coming from both holes, it is possible to assume that the resulting distribution is a linear combination of two Gaussians. This explains the distribution on the backstop when both holes are open.

The two-slit experiment performed with elementary particles falsifies the conjunction of all assumptions we have made. In other words, the fact that in the case where these particle frequencies interfere does not only mean that the axioms of probability do not hold for them. It means that these axioms together with all hypotheses we have assumed in order to explain the bullets' frequency do not hold for microscopic objects. I strongly feel that the weak part of this set of assumptions is not the addition rule, but the various types of independence on which the theory of errors is based.

3. In order to make clear what I have in mind, a few words may be useful. First of all, I shall carry on with my considerations in the direction pointed out by Feynman, i.e., I take elementary particles as the object of my analysis as well. Hence I shall take into account random variables intended to denote particles of unspecified nature, but I make the experiment even more simple than usual. I consider only three particles and two sites on the backstop. I am interested in which hole the particles go through, and from which site they are absorbed. As a consequence, I consider three random variables, X_1, X_2, and X_3, subscripts denoting the order in which particles come from a source, say a. These variables can assume one of the two attributes of a first family $H=\{h_1, h_2\}$ denoting holes, and one of the two attributes of a second family $S=\{s_1, s_2\}$ denoting sites. That is, each random variable can assume as value an ordered pair (h_j, s_g), $j=1, 2$ and $g=1, 2$. Hence $X_i=(h_j, s_g)$ asserts that the ith particle has gone through the jth hole and has been absorbed from the gth site of the backstop. For the sake of simplicity, I shall write this as a conjunction, i.e., $X_i=h_j \,\&\, X_i=s_g$.

The state (of individuals) descriptions (possibilities) relative to holes are

$$X_1=h_1 \,\&\, X_2=h_1 \,\&\, X_3=h_1, \qquad X_1=h_2 \,\&\, X_2=h_1 \,\&\, X_3=h_1, \ldots,$$
$$X_1=h_2 \,\&\, X_2=h_2 \,\&\, X_3=h_2$$

Those relative to sites are

$$X_1=s_1 \,\&\, X_2=s_1 \,\&\, X_3=s_1, \qquad X_1=s_2 \,\&\, X_2=s_1 \,\&\, X_3=x_1, \ldots,$$
$$X_1=s_2 \,\&\, X_2=s_2 \,\&\, X_3=s_2$$

These are not the possibilities I am interested in. We must consider composite possibilities, that is, we must go down and up from H to S and vice versa. The result are 64 Q-state descriptions, with the first being

$$\mathfrak{Q}\mathfrak{Z}_1 := X_1=h_1 \,\&\, X_1=s_1 \,\&\, X_2=h_1 \,\&\, X_2=s_1 \,\&\, X_3=h_1 \,\&\, X_3=s_1$$

i.e., $\mathfrak{Q}\mathfrak{Z}_1$ describes the possibility according to which all the three particles go through hole 1 and are absorbed from site 1.

However, we are not only interested in individual descriptions, but we are also interested in Q-structure descriptions, i.e., in the number of particles which go through holes and are absorbed from the backstop. Obviously each Q-structure description is a (possible) frequency distribution. One of these is

$$\mathfrak{Q}\mathfrak{S}_1 := (3, 0; 3, 0)$$

It says that all particles have gone through h_1 and have been absorbed from s_1. Obviously $\mathfrak{Q}\mathfrak{Z}_1$ is the sole individual description which belongs to $\mathfrak{Q}\mathfrak{S}_1$, but in general to a Q-structure description there correspond more than one Q-state descriptions.

If we are only interested in the number of particles that have been absorbed from the backstop, we can restrict our attention to the structure description for sites. One of these descriptions is

$$\mathfrak{S}_1 := (3, 0).$$

$\mathfrak{Q}\mathfrak{S}_1$ belongs to $\mathfrak{S}_1$, but there are other Q-structure descriptions belonging to $\mathfrak{S}_1$.

The distributions of particles on the backstop are described by structure descriptions. It is clear than when we know the probabilities of all Q-state descriptions, we can determine the probabilities of structure descriptions. This is the reason why I am limiting my attention to individual descriptions.

Even if it is often forgotten, the product rule is also a law of classical probability theory. Making use of this rule, one can give in full detail the probability of $\mathfrak{Q}\mathfrak{Z}_1$ as follows:

$$\begin{aligned}\Pr\{\mathfrak{Q}\mathfrak{Z}_1\} = {} & \Pr\{X_1 = h_1 | X_1 = a\} \Pr\{X_1 = s_1 | X_1 = a, X_1 = h_1\} \\ & \times \Pr\{X_2 = h_1 | X_1 = a, X_1 = h_1, X_1 = s_1, X_2 = a\} \\ & \times \Pr\{X_2 = s_1 | X_1 = a, X_1 = h_1, X_1 = s_1, X_2 = a, X_2 = h_1\} \\ & \times \Pr\{X_3 = h_1 | X_1 = a, X_1 = h_1, X_1 = s_1, X_2 = a, X_2 = h_1, \\ & \qquad X_2 = s_1, X_3 = a\} \\ & \times \Pr\{X_3 = s_1 | X_1 = a, X_1 = h_1, X_1 = s_1, X_2 = a, X_2 = h_1, \\ & \qquad X_2 = s_1, X_3 = a, X_3 = h_1\} \end{aligned} \tag{1}$$

where $X_i = a$ asserts that particle X_i comes from the source a.

First I will consider a case analogous to that of bullets. It is useful to keep in mind that, for example, by the abridgement "dependence upon hole" I intend "dependence upon the hole in which the particle and the previous ones have gone through." Assuming independence on source, holes, and sites, we have

$$\begin{aligned}\Pr\{\mathfrak{Q}\mathfrak{Z}_1\} = {} & \Pr\{X_1 = h_1 | X_1 = a\} \Pr\{X_1 = s_1 | X_1 = a\} \Pr\{X_2 = h_1 | X_2 = a\} \\ & \times \Pr\{X_2 = s_1 | X_2 = a\} \Pr\{X_3 = h_1 | X_3 = a\} \Pr\{X_3 = s_1 | X_3 = a\}\end{aligned}$$

Putting $\Pr\{X_i = h_1 | X_i = a\} = \Pr\{X_i = s_1 | X_i = a\} = 1/2$, $i = 1$, 2, and 3, the probability of each Q-state description becomes equal to 1/64. It is easy to check that the description on the backstop when both holes are open is equal to the sum of both with weights 1/2 when each of the two holes is open alone.

The particular case of (1) we have just considered is one of the possible types of independence we can suppose to exist among particles. Always referring to $\mathfrak{Q}\mathfrak{z}_1$ and as a sort of exercise, I will now consider some cases in which one or more types of dependence are dropped.

(i) Only hole 1 open; dependence upon source and site:

$$\Pr\{\mathfrak{D}\mathfrak{z}_1\} = \Pr\{X_1 = s_1 | X_1 = a\} \Pr\{X_2 = s_1 | X_1 = a, X_1 = s_1, X_2 = a\}$$
$$\times \Pr\{X_3 = s_1 | X_1 = a, X_1 = s_1, X_2 = a, X_2 = s_1, X_3 = a\}$$

Notice that this is also the probability of $\mathfrak{Q}\mathfrak{z}_1$ in the case when, notwithstanding both holes are open, we know that all particles have gone through hole 1. In fact, in this case

$$\Pr\{X_i = h_1 | E\} = 1, \qquad i = 1, 2, \text{ and } 3; \quad E \text{ whatsoever}$$

(ii) Only hole 1 open; dependence upon source of the considered particle:

$$\Pr\{\mathfrak{Q}\mathfrak{z}_1\} = \Pr\{X_1 = s_1 | X_1 = a\} \Pr\{X_2 = s_1 | X_2 = a\} \Pr\{X_3 = s_1 | X_3 = a\}$$

(iii) Both holes open; dependence upon source and hole:

$$\Pr(\mathfrak{Q}\mathfrak{z}_1) = \Pr\{X_1 = h_1 | X_1 = a\} \Pr\{X_1 = s_1 | X_1 = a, X_1 = h_1\}$$
$$\times \Pr\{X_2 = h_1 | X_1 = a, X_1 = h_1, X_2 = a\}$$
$$\times \Pr\{X_2 = s_1 | X_1 = a, X_1 = h_1, X_2 = a, X_2 = h_1\}$$
$$\times \Pr\{X_3 = h_1 | X_1 = a, X_1 = h_1, X_2 = a, X_2 = h_1, X_3 = a\}$$
$$\times \Pr\{X_3 = s_1 | X_1 = a, X_1 = h_1, X_2 = a, X_2 = h_1, X_3 = a, X_3 = h_1\}$$

(iv) Both holes open; dependence upon source and site:

$$\Pr\{\mathfrak{Q}\mathfrak{z}_1\} = \Pr\{X_1 = h_1 | X_1 = a\} \Pr\{X_1 = s_1 | X_1 = a\}$$
$$\times \Pr\{X_2 = h_1 | X_1 = a, X_1 = s_1, X_2 = a\}$$
$$\times \Pr\{X_2 = s_1 | X_1 = a, X_1 = s_1, X_2 = a\}$$
$$\times \Pr\{X_3 = h_1 | X_1 = a, X_1 = s_1, X_2 = a, X_2 = s_1, X_3 = a\}$$
$$\times \Pr\{X_3 = s_1 | X_1 = a, X_1 = s_1, X_2 = a, X_2 = s_1, X_3 = a\}$$

(v) Both holes open; dependence upon source and hole; hole destroys dependence upon source:

$$\begin{aligned}\Pr\{\mathfrak{Q}\mathfrak{Z}_1\} = {} & \Pr\{X_1 = h_1 | X_1 = a\} \Pr\{X_1 = s_1 | X_1 = h_1\} \\ & \times \Pr\{X_2 = h_1 | X_1 = h_1, X_1 = s_1, X_2 = a\} \\ & \times \Pr\{X_2 = s_1 | X_1 = h_1, X_1 = s_1, X_2 = h_1\} \\ & \times \Pr\{X_3 = h_1 | X_1 = h_1, X_1 = s_1, X_2 = h_1, X_2 = s_1, X_3 = a\} \\ & \times \Pr\{X_3 = s_1 | X_1 = h_1, X_1 = s_1, X_2 = h_1, X_2 = s_1, X_3 = h_1\}\end{aligned}$$

This last case seems to have something in common with the usual approach to quantum probability.

4. The above examples make clear the sense in which the usual analysis of the two-slit experiment is too rough. But there is something even more significant. In principle we cannot exclude that in the case of microscopic objects the distribution on the backstop may be reached by a suitable use of dependences affecting probabilities of Q-state descriptions involved in the experiment. A simple example will clarify the point.

In a famous physical textbook, Feynman *et al.* (1965) consider two not identical Bose particles, x and y scattered from two different scatterers into two states 1 and 2 which are nearly the same. $\langle j|i\rangle$ is the amplitude that particle i is scattered into state j. The amplitude for the two scatterings, x into 1 and y into 2, and the related probability are, respectively, $\langle 1|x\rangle\langle 2|y\rangle$ and $|\langle 1|x\rangle|^2 |\langle 2|y\rangle|^2$. The amplitude for x into 2 and y into 1 and the related probability are, respectively, $\langle 2|x\rangle\langle 1|y\rangle$ and $|\langle 2|x\rangle|^2 |\langle 1|y\rangle|^2$. At this point Feynman *et al.* (1965, p. 4-4) say, "Imagine now we have a pair of tiny counters that pick up the two scattered particles. The probability P_2 that they will pick up two particles together is just the sum" $P_2 = |\langle 1|x\rangle|^2 |\langle 2|y\rangle|^2 + |\langle 2|x\rangle|^2 |\langle 1|y\rangle|^2$. When the two states are close enough, the two amplitudes $\langle 1|i\rangle$ and $\langle 2|i\rangle$ will be equal to $\langle s|i\rangle$. As a result we get $P_2 = 2|\langle s|x\rangle|^2 |\langle s|y\rangle|^2 = \Pr\{x = s\} \Pr\{y = s\}$. Then the authors consider two identical Bose particles. In this case the amplitude for the two different scatterings can interfere. As a consequence the total amplitude of getting a particle in each counter is $\langle 1|x\rangle\langle 2|y\rangle + \langle 2|x\rangle\langle 1|y\rangle$. Again, if the states are close enough, the probability is the absolute square of this amplitude, that is

$$P_2 = |\langle 1|x\rangle\langle 2|y\rangle + \langle 2|x\rangle\langle 1|y\rangle|^2 = 4|\langle s|x\rangle|^2 |\langle s|y\rangle|^2 = 2 \Pr\{x = s\} \Pr\{y = s\}$$

Feyman *et al.* remark, "We have the result that it is twice as likely to find two *identical* Bose particles scattered into the same state *as you would calculate assuming the particles were different*" (Feynman *et al.*, 1965, p. 4-4).

Once again the above statistical analysis is not accurate. Feynman *et al.* implicitly maintain that it is impossible to calculate the probability for two identical Bose particles to be in the same state without making reference to amplitudes. Resorting to this notion solves the problem, showing that the wrong value calculated with classical probabilities is half the right one calculated with quantum probabilities. Of course this second assertion is right, but an accurate statistical analysis gives the right value for classical probability as well. The equality $\Pr\{x = s \,\&\, y = s\} = \Pr\{x = s\} \Pr\{y = s\}$ only holds if $x = s$ and $y = s$ are independent. If this is not the case, then $\Pr\{x = s \,\&\, y = s\} = \Pr\{x = s\} \Pr\{y = s | x = s\}$ holds. This is what I have already stressed in Section 3, but there is more to come. Nobody can exclude that $\Pr\{y = s | x = s\} = 2 \Pr\{y = s\}$ for the simple reason that it is really the case. In fact, assuming some suitable conditions on the probability function, it is possible to show that (i) $\Pr\{y = s\} = k^{-1}$ and (ii) $\Pr\{y = s | x = s\} = 2(k + 1)^{-1}$, where k is the number of oscillators (states) (Costantini and Garibaldi, 1986). That is, in the case in which $k \gg 1$, $\Pr\{y = s | x = s\}$ is almost twice $\Pr\{y = s\}$. Clearly this holds when $x = s$ and $y = s$ are positively correlated and the correlation has a well-fixed value.

However, this is not all. It is also possible to show that "the probability of counting n Bose particles together is $n!$ greater than we would calculate assuming that the particles were all distinguishable" (Feynman *et al.*, 1965, p. 4-7) and "the probability of getting a boson, where there are already n, is $(n + 1)$ times stronger than it would be if there were none before" (Feynman *et al.*, 1964, p. 4-7). More precisely, it is possible to prove that the probability of counting n independent particles on the same oscillators is k^{-n} and that the probability of counting n Bose particles together is

$$\frac{n!}{k(k+1) \cdots (k+n-1)}$$

The ratio of these probabilities is

$$n! \frac{k}{k} \frac{k}{k+1} \cdots \frac{k}{k+n-1}$$

Again if the number of oscillators is much greater than that of the particles, this ratio is equal to $n!$.[2]

What is worth noting in this example is that there is at least an eventuality in which classical probability and dependence give the same

[2]Of course, in considering a great number of states we must exercise more caution. For the derivation of this case, i.e., for the derivation of the grand canonical distribution for bosons, fermions, and classical particles, see Costantini and Garibaldi (n.d.).

result as quantum probability. Going back to the two-slit experiment, the probability of each Q-state description involved in that experiment can be determined when we are able to determine the value of

$$\Pr\{X_i = j|E\} \tag{2}$$

where E is an evidence whatsoever, but this proves to be a hard task. Due to the fact that attributes of two families appear in E, the determination of (2) is a challenge. Nevertheless, we cannot exclude the possibility, at least in principle, that a suitable allotment of value to the probabilities of type (2) that are components of the composite probability of a Q-state description can give rise to the frequency distribution observed to the backstop in the two-slit experiment.

5. What I have said authorizes us to entertain the conviction that there is only one notion of probability applicable to both the macroscopic and the microscopic world. That is, the difference between classical and quantum probability stems from a lot of new probability rules discovered by physicists during the study of elementary particles. In other words, with Maxwell, Boltzmann, Einstein, Schrödinger and Dirac, the notion of probability was used in a new sphere. This allowed the discovery of a number of new statistical methods related to stochastic dependence. Contrary to what happened in other similar cases, this fact has not yet been recognized by the scientific community, whose members prefer to refer to a new concept, i.e., quantum probability.

In the work on which my conviction is based (Costantini and Garibaldi, 1990, 1993, n.d.), great importance is given to relative (transition) probabilities. Contrary to the absolute probabilities which are generally used in physics, relative probabilities enable an immediate understanding of dependence. That is, relative probabilities face correlation directly, and what is more important, make clear on what type of notions correlations apply. However, this is the case only if the status of the probability functions is well defined. The transition probabilities we have worked with pertain to the change of state on an oscillator, that is, to the probability with which a particle accommodates on an oscillator when the structural description of the system of oscillators is well specified. By applying these probabilities, we can determine the finite-dimensional distributions of the stochastic process describing the growth of the total energy of the system. This enables the determination of the familiar elementary-particle statistics which are probability distributions on possible populations (of particles). The thermodynamic limit allows infinity to enter the picture. This makes oscillators independent, but it does not destroy the dependence of particles. As a result we have a probability distribution on possible states of oscillators which again is a distribution on possible populations. Considering various types of excitations, entropy indicates the most probable combination, together with Planck's radiation formula. Hence in every stage of our derivation, the status of each probability distribution we are considering is well defined.

This remains true when we try to interpret the result of the derivation considering it as the sampling distribution of a test of significance. That is, in order to check Planck's radiation formula, we must sample the number of bosons per oscillator. That is, we should take different values of the occupation number per oscillator in independent trials and determine the sampling distribution of that number. Nevertheless, this does not take into account the enormous velocity at which change occurs at the macroscopic level. Due to the fact that the time resolution of the analyzer is much greater than the characteristic fluctuation time, an experimental value is actually a mean value. It is a value of the (in practice deterministic) function giving the mean occupation number of an oscillator. In this manner it is possible to make precise what type of correlation produces the radiation coming out from a furnace.

6. In quantum mechanics we do not find such accuracy, as is easy to realize taking into account a simple textbook problem (Feynman *et al.*, 1965, Chapter 8). The example is related to the treatment of the states of an ammonia molecule, which essentially amounts to the derivation of Pauli spin matrices. The two possible positions of the nitrogen atom are taken as the two states $|1\rangle$ and $|2\rangle$ of the molecule. As is well known, the actual state of the molecule ψ can be represented by the amplitude to be in $|1\rangle$, i.e., $C_1 = \langle 1|\psi\rangle$, and that to be in $|2\rangle$, i.e., $C_2 = \langle 2|\psi\rangle$. Now I take into account the way in which the state ψ varies with time, i.e.,

$$\frac{ih}{\pi}\frac{dC_1}{dt} = H_{11}C_1 + H_{12}C_2$$

$$\frac{ih}{\pi}\frac{dC_2}{dt} = H_{21}C_1 + H_{22}C_2$$

where h is Planck's constant. The H_{ij} coefficients are the elements of the Hamiltonian matrix that, considering an infinitesimal interval of time, I will write as transition amplitudes using the notation of the preceding sections, but putting A (for amplitude) instead of Pr (for probability). Hence

$$H_{ij} = \mathrm{A}\{X_{t+\Delta t} = i | X_t = j\}$$

where $X_{t+\Delta t}$ is a random variable describing the ammonia molecule at a time $t + \Delta t$, while X_t is a random variable describing the ammonia molecule at a time t. These random variables can assume only two values, 1 and 2, corresponding to states $|1\rangle$ and $|2\rangle$. From the amplitudes we can get probabilities:

$$\Pr\{X_{t+\Delta t} = i | X_t = j\} = |\mathrm{A}\{X_{t+\Delta t} = i | X_t = j\}|^2$$

Having fixed the base states and limiting our attention to them, each hypothesis upon amplitudes is also a hypothesis upon probabilities. Bearing this in mind, it is possible to comment on the way in which the values of the Hamiltonian matrix are determined. This determination is possible thanks to two special hypotheses.

The first hypothesis, which I call **A1**, is as follows: Supposing "that once the molecule was in the state $|1\rangle$, there was no chance that it would ever get into $|2\rangle$ and vice versa (Feynman *et al.*, 1965, p. 8-12), H_{12} and H_{21} would both be equal to 0. If this is the case, considering that "for the ammonia molecule the two states $|1\rangle$ and $|2\rangle$ have definite symmetry. If nature is at all reasonable, the matrix elements H_{11} and H_{12} must be equal" (Feynman *et al.*, 1965, p. 8-12).

The second hypothesis, which I call **A2**, is as follows: After having noted that there is some amplitude that the nitrogen will penetrate the energy barrier of the three hydrogen atoms, the authors assert, "The coefficients H_{12} and H_{21} are not really zero. Again, by symmetry, they should both be the same—at least in magnitude" (Feynman *et al.*, 1965, p. 8-12).

There are no doubts that these are probabilistic hypotheses. H_{jj} is related to an initial transition, where initial is intended to stress the fact that the considered transition does not change the state. **A1** imposes the equality of initial transition probabilities. In deriving the result recalled in Section 4, we must consider initial probabilities and impose the equality of such probabilities. Regarding a given oscillator, the initial probability is the probability with which a particle accommodates on the oscillator when the system is in the fundamental state. Initial probabilities and initial transition probabilities have someting in common. For this it is worth considering the status of initial probabilities. In general, a condition imposing the equidistribution of the initial probabilities amount to stating

$$\text{for each } i, \qquad \Pr\{X_i = j\} = k^{-1}, \qquad j = 1, \ldots, k$$

that is, the probability that any individual whatsoever bears an attribute of a family, if nothing is known about the population, is equal to the inverse of the cardinality of the family. Applied to elementary particles, such a condition imposes that each initial probability be equal to the inverse of the number of oscillators: more explicitly, that the probability to go from the fundamental to the first excited state when the system is in the fundamental state is the same for all oscillators. What is worth stressing is that using this condition, we are completely clear with respect to which entities we are speaking about.

This is not the case for **A1**. About what is this hypothesis speaking? Does it assert something on an ammonia molecule? Or does it assert something about the population of all ammonia molecules? Someone could answer that this is a problem of interpretation and that such problems do not pertain to formal theories, but this is not a problem of interpretation, as easily shown by considering **A2**. This hypothesis imposes a probabilistic property to transitions which are not initial. More exactly, it asserts that the amplitude of the transition from $|1\rangle$ to $|2\rangle$ is equal to that from $|2\rangle$ to $|1\rangle$. Yet, again, what exactly is **A2** speaking about? Is it the transition of an ammonia molecule or the transition of a population of such molecules?

The difficulty I have in mind is the following: **A1** and **A2** are conditions relating to states. But states of what? Oscillators may be attributes of particles, but (numbers of) particles may be attributes of oscillators. It is not difficult to find in the literature these different meanings of the term "state." For example, as we have seen in Section 4, Feynman *et al.* speak of the oscillator as the state of a particle, but in another occasion Feynman (1988, Section 1.2) speaks of the number of bosons as the state of an oscillator. It follows that one may refer to the probability that a particle "bears" an oscillator, but also to the probability that an oscillator "bears" m particles. In the first case, we are considering the probability that a particle has a property; whereas in the second we are speaking of the probability that a population of particles has a property. In the first case, one is considering an individual probability; on the contrary, in the second, one is considering a universal probability. From a probabilistic point of view, this difference is far from being trivial.

7. I have shown that "site" correlation can be taken into account only when we are able to formulate hypotheses relative to the probabilistic behavior of (generic) particles. On the one hand, I am inclined to think that the same holds for the correlation produced from both "hole" and "site" together. On the other hand, I suspect that random variables used in quantum mechanics speak about populations. If this is the case, the behavior of relative frequencies in the population can darken the correlation of its members. In other words, a universal probability may hide the peculiar properties of the individual probabilities from which it has been derived. The derivation of Planck's formula makes it clear how it can happen. In fact, the blackbody radiation spectrum does not show any interference, but in spite of this it is determined from the boson correlation.

The consequences of the lack of clarity regarding entities on which we are working are far-reaching. The interference we note in the two-slit experiment and similar phenomena are properties of frequency distributions. That is, they are properties of populations, not of particles. The interference in the two-slit experiment is produced by "hole" and "site" correlation. Undoubtedly this interference is a phenomenon whose complexity is much greater than correlation of the blackbody radiation. The correlation ruling interferences of this type is surely of a more complex form than that of the bosons coming out of a furnace, but it is a correlation related to particles whose effects rule the behavior of the related frequency distributions. The worst of this is that the original particle correlation can be completely distorted by frequencies, i.e., from the way in which particles accommodate on "sites." If this is the case, then in order to clarify frequency interferences, we must look at particles, not at populations.

This assertion deserves some comment. When speaking of events H and H', we say that probability is an additive function, i.e.,

$$\text{if} \quad E \cap H \cap H' = \varnothing, \qquad \Pr\{H \cup H' | E\} = \Pr\{H|E\} + \Pr\{H'|E\}$$

we are indeed making a lot of assertions. Considering only those more familiar, we are speaking of attributes of a family, sets, relative frequencies, and individuals (generic or specific). Of course, this is the great advantage of the formal approach to theories. But when we want to interpret our language, we must define exactly what we are speaking about.

The superposition of pure states is a principle whose nature is similar to that of the addition principle of the classical probability theory. Via amplitudes, the superposition principle introduced dependence into classical probability theory. Like the addition principle, it works very well. Using the superposition principle, physicists have produced a great triumph of contemporary physics. The results coming from the applications have shown without any doubts its utility in increasing our knowledge of the microscopic world, but as in the case of the addition principle, using the superposition principle, we do not know exactly what we are speaking about. More precisely, in using this principle, are we speaking of specific individuals, generic ones, attributes, relative frequencies, or something else? I am inclined to think that the foundational difficulties related to quantum correlation arise from the ambiguity of the probability function which is connected with the superposition principle. I am convinced that if we want to clarify the role of quantum dependence, we should be more precise. We must work out the probability conditions which rule the statistical behavior of elementary particles.

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